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Compact Imbeddings of Weighted Sobolev Spaces on Unbounded Domains*

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1. INTRODUCTION

Let G be an unbounded open set in Euclidean n-space, E_n. We denote by $W_0^{m,p,\lambda}(G)$ the completion of the space $C_0^\infty(G)$ of infinitely differentiable functions with compact support in G, in the norm

$$\| u \|_{m,p,\lambda} = \left\{ \| u \|_{p,G}^p + \sum_{|\alpha|=m} \int_G | D^\alpha u(x)|^p (1 + | x |)^\lambda \, dx \right\}^{1/p},$$

where $\| u \|_{p,G}$ is the norm in $L^p(G)$. Here λ is any real number, m is a positive integer, and $p \geqslant 1$; $\alpha = (\alpha_1, ..., \alpha_n)$ is an n-tuple of nonnegative integers; $| \alpha | = \sum \alpha_i$; $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$ where $D_j = \partial/\partial x_j$.

In this paper we characterize, for each λ, those domains G for which the natural imbedding

$$W_0^{m,p,\lambda}(G) \to L^p(G)$$

is completely continuous. For $\lambda > mp$ this is the case for all G. If $\lambda \leqslant mp$ a necessary and sufficient condition on G to guarantee the compactness of the imbedding is formulated in terms of a certain definition of capacity for closed subsets of cubes in E_n. A geometric condition on G which implies this capacity condition is then given. These results generalize similar results obtained by the author in Ref. [1] for the special case $\lambda = 0$. Finally we illustrate an implication of the compactness theorems for partial differential operators in G having degenerate or divergent ellipticity at infinity.

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2. Characterization of Compact Imbeddings

We begin by disposing of the case $\lambda > mp$.

THEOREM 1. *If $\lambda > mp$ the imbedding $W_0^{m,p,\lambda}(G) \to L^p(G)$ is compact for every open $G \subset E_n$.*

Proof. If the imbedding is compact for G it is compact for any $G' \subset G$ so without loss of generality we take $G = E_n$. If $u \in C_0^\infty(E_n)$ we have

$$u(r, \sigma) = \frac{(-1)^m}{(m-1)!} \int_r^\infty (t-r)^{m-1} \frac{d^m}{dt^m} u(t, \sigma)\, dt,$$

where (r, σ) denote spherical polar coordinates in E_n. If $p > 1$ it follows from an application of Holder's inequality that

$$\begin{aligned} | u(r, \sigma)|^p r^{n-1} &\leqslant \frac{1}{(m-1)!} \int_r^\infty \Big| \frac{d^m}{dt^m} u(t, \sigma)\Big|^p t^{n-1}(1+t)^\lambda\, dt \\ &\quad \times \Big[\int_r^\infty (t-r)^{(mp-p)/(p-1)}(1+t)^{-\lambda/(p-1)}\, dt\Big]^{p-1} \\ &\leqslant \text{const} \cdot r^{mp-\lambda-1} \int_r^\infty \Big| \frac{d^m}{dt^m} u(t, \sigma)\Big|^p t^{n-1}(1+t)^\lambda\, dt, \end{aligned}$$

where the constant depends on m, p, and λ. If $p = 1$ a similar inequality is immediate.

Let $K_R = \{x \in E_n : | x | \geqslant R\}$. If Σ denotes the unit sphere in E_n we obtain

$$\begin{aligned} \| u \|_{p,K_R}^p &\leqslant \text{const} \cdot \int_R^\infty r^{mp-\lambda-1}\, dr \int_\Sigma d\sigma \int_0^\infty \Big| \frac{d^m}{dt^m} u(t, \sigma)\Big|^p t^{n-1}(1+t)^\lambda\, dt \\ &\leqslant \text{const} \cdot R^{mp-\lambda} \| u \|_{m,p,\lambda}^p \end{aligned}$$

and this inequality holds, by completion, for all u in $W_0^{m,p,\lambda}(E_n)$. Let $\{u_i\}_{i=1}^\infty$ be a bounded sequence in $W_0^{m,p,\lambda}(E_n)$. In order to show that $\{u_i\}$ is precompact in $L^p(E_n)$ it is sufficient to show that

(a) For every $\epsilon > 0$ there exists $R \geqslant 0$ such that $\| u_i \|_{p,K_R} < \epsilon$ for all i;

(b) For any bounded G the sequence $\{u_i \mid G\}$ is precompact in $L^p(G)$.

(a) is an immediate consequence of the inequality obtained above. To establish (b) note that $G \subset B$ for some ball B. Since $(1 + | x |)^{-\lambda}$ is bounded on B it follows that $\{u_i \mid B\}$ is bounded in the ordinary Sobolev space $W^{m,p}(B)$ and hence precompact in $L^p(B)$ by Kondrachov's theorem. Thus $\{u_i \mid G\}$ is compact in $L^p(G)$ and the theorem follows.

We turn now to the case $mp \geqslant \lambda$. If H is a cube of side h in E_n and if E is a closed proper subset of H we denote by $C^\infty(H, E)$ the class of all functions in $C^\infty(H)$ which vanish near E. We define the $W^{m,p}$ capacity of E in H as

$$C_{H,E}^{m,p} = \inf_{u \in C^\infty(H,E)} \frac{I_H^{m,p}(u)}{\| u \|_{p,H}^p},$$

where

$$I_H^{m,p}(u) = \sum_{1 \leqslant |\alpha| \leqslant m} h^{p|\alpha|} \int_H | D^\alpha u(x)|^p \, dx.$$

All cubes H referred to in this paper will have faces parallel to the coordinate planes.

THEOREM 2. *If $\lambda \leqslant mp$ then necessary and sufficient for the compactness of the imbedding $W_0^{m,p,\lambda}(G) \to L^p(G)$ is the following condition on G: For every $\epsilon > 0$ there exist R, s with $R \geqslant 0$ and $0 < s \leqslant 1$, and a partition X of E_n into cubes H of side h satisfying*

$$h^{mp} \leqslant s^m \inf_H (1 + | x |)^\lambda$$

such that $C_{H,H-G}^{m,p} \geqslant s/\epsilon$ for each $H \in X$ for which $H \cap G_R$ is nonempty, G_R being $\{x \in G : | x | \geqslant R\}$.

Before proving Theorem 2 we prepare the following lemmas.

LEMMA 1. *There is a constant K_1 such that for every cube H of side h in E_n and every $u \in C^\infty(H)$*

$$\sum_{|\alpha|=0}^{m} h^{p|\alpha|} \| D^\alpha u \|_{p,H}^p \leqslant K_1 \left[\| u \|_{p,H}^p + \sum_{|\alpha|=m} h^{mp} \| D^\alpha u \|_{p,H}^p \right].$$

Proof. For $H = H_1$, the cube of unit side centred at the origin, the above inequality is well-known (cf. Agmon, Ref. [2, Theorem 3.3] for the case $p = 2$). For general H of side h and centre c, if $u \in C^\infty(H)$ then $u(x) = v(y)$, where $y = (x - c)/h$ for some $v \in C^\infty(H_1)$, and the lemma follows by change of variables.

LEMMA 2. *There exists a constant K_2 such that for any $\epsilon > 0$ there exists $\delta = \delta(\epsilon)$ with $0 < \delta < 1$ such that for any cube H of side h in E_n and any $u \in C^\infty(H)$*

$$\| u \|_{p,H}^p \leqslant \epsilon h^{mp} \sum_{|\alpha|=m} \| D^\alpha u \|_{p,H}^p + K_2 \| u \|_{p,\delta H}^p,$$

where δH is the cube of side δh concentric with H.

Proof. By another standard interpolation theorem (cf. Agmon, Ref. [2, Theorem 3.2(2°)] for the case $p = 2$) there exists a constant K_2 such that for any $\epsilon > 0$ there exists $\delta' = \delta'(\epsilon) < 1$ such that

$$\| v \|_{p,H_1}^p \leqslant \frac{K_2}{2} [\epsilon \| \operatorname{grad} v \|_{p,H_1}^p + \| v \|_{p,\delta' H_1}^p]$$

$$\leqslant \frac{K_1 K_2 \epsilon}{2} \Big[\| v \|_{p,H_1}^p + \sum_{|\alpha|=m} \| D^\alpha v \|_{p,H_1}^p\Big] + \frac{K_2}{2} \| v \|_{p,\delta' H_1}^p$$

for every $v \in C^\infty(H_1)$. Putting $\delta(\epsilon) = \delta'(\epsilon/K_1 K_2)$ we obtain

$$\| v \|_{p,H_1}^p \leqslant \epsilon \sum_{|\alpha|=m} \| D^\alpha v \|_{p,H_1}^p + K_2 \| v \|_{p,\delta H_1}^p$$

for $\epsilon \leqslant 1$ and hence for any ϵ. The case of general H follows just as in Lemma 1.

LEMMA 3. *For given δ, $0 < \delta < 1$, there exists a constant K_3 such that for any cube H of side h in E_n there exists $w \in C_0^\infty(H)$ with the properties*

(a) $w(x) = 1$ *for* $x \in \delta H$;

(b) $| D^\alpha w(x)| \leqslant K_3 h^{-|\alpha|}$ *for all* x, $0 \leqslant | \alpha | \leqslant m$.

Once again the proof follows from the special case $H = H_1$ by change of variable.

LEMMA 4. *If $-\infty < \lambda \leqslant mp$ there exists a positive constant c and a partition X of E_n into cubes H of side h satisfying*

$$c \sup_H (1 + | x |)^\lambda \leqslant h^{mp} \leqslant \inf_H (1 + | x |)^\lambda$$

Proof. Construct the cubic block centred at the origin consisting of 3^n cubes of side 1. Surround this block with a layer of cubes of side 3 to make a new cubic block of volume 3^n times that of the old block. Continue adding layers of cubes in this way, trippling the cube edge each time a new layer is added. It is easily verified that each cube H' of the partition X' so generated satisfies, h' being the side of H',

$$c' \sup_{H'} (1 + | x |) \leqslant h' \leqslant k' \inf_{H'} (1 + | x |)$$

(in fact $c' = 1/3 \sqrt{n}$, $k' = 2$). Since $mp \geqslant \lambda$ we have for each cube H' in the partition

$$c'^{mp} \sup_{H'} (1 + | x |)^\lambda \leqslant c'^{mp} \sup_{H'} (1 + | x |)^{mp} \leqslant h'^{mp}.$$

If also $h'^{mp} \leqslant \inf_{H'}(1 + |x|)^\lambda$ let $H = H'$, $h = h'$. Otherwise subdivide H' into N^n cubes H of side $h = h'/N$ where N is the smallest integer such that $h^{mp} \leqslant \inf_{H'}(1 + |x|)^\lambda$. It follows that

$$h^{mp} \geqslant \left(\frac{N-1}{N}\right)^{mp} \inf_{H'} (1 + |x|)^\lambda \geqslant (1/2)^{mp} (c'/k')^\lambda \sup_{H'} (1 + |x|)^\lambda.$$

Subdividing $H' \in X'$ in this manner whenever necessary we obtain a new partition X for all cubes H of which we have

$$c \sup_H(1 + |x|)^\lambda \leqslant h^{mp} \leqslant \inf_H(1 + |x|)^\lambda,$$

where $c = \min(c'^{mp}, (1/2)^{mp}(c'/k')^\lambda)$.

LEMMA 5 (Interpolation Inequality for Weighted Spaces). *If $mp \geqslant \lambda$ there exists a constant K_0 such that for any $u \in C_0^\infty(E_n)$.*

$$\sum_{0 \leqslant |\alpha| \leqslant m} \int_{E_n} | D^\alpha u(x)|^p (1 + |x|)^{\lambda|\alpha|/m} \, dx$$

$$\leqslant K_0 \left\{\| u \|_{p,E_n}^p + \sum_{|\alpha|=m} \int_{E_n} | D^\alpha u(x)|^p (1 + |x|)^\lambda \, dx\right\}.$$

Proof. Let X be the partition constructed in Lemma 4. Let

$$c_0 = \max_{0 \leqslant |\alpha| \leqslant m} c^{-|\alpha|/m} > 0.$$

By Lemma 1

$$\sum_{0 \leqslant |\alpha| \leqslant m} \int_H | D^\alpha u(x)|^p (1 + |x|)^{\lambda|\alpha|/m} \, dx$$

$$\leqslant c_0 \sum_{0 \leqslant |\alpha| \leqslant m} h^{p|\alpha|} \int_H | D^\alpha u(x)|^p \, dx$$

$$\leqslant K_1 c_0 \left[\| u \|_{p,H}^p + h^{mp} \sum_{|\alpha|=m} \int_H | D^\alpha u(x)|^p \, dx\right]$$

$$\leqslant K_0 \left[\| u \|_{p,H}^p + \sum_{|\alpha|=m} \int_H | D^\alpha u(x)|^p (1 + |x|)^\lambda \, dx\right].$$

The lemma now follows by summation over the finite number of cubes $H \in X$ which meet the support of u.

Proof of Theorem 2 (Sufficiency). Let $\epsilon > 0$ and suppose there exists $R \geqslant 0$ and s, $0 < s \leqslant 1$, and a partition X of E_n into cubes H of side h

satisfying $h^{mp} \leqslant s^m \inf_H(1 + |x|)^\lambda$ such that $C^{m,p}_{H,H-G} \geqslant sK_0/\epsilon^p$ for every H in X which meets G_R. Then for any such H and for any $u \in C_0^\infty(G)$ we have, noting that $s^{|\alpha|} \leqslant s$,

$$\| u \|^p_{p,H} \leqslant \frac{\epsilon^p}{sK_0} \sum_{1\leqslant|\alpha|\leqslant m} h^{p|\alpha|} \int_H | D^\alpha u(x)|^p \, dx$$

$$\leqslant \frac{\epsilon^p}{K_0} \sum_{1\leqslant|\alpha|\leqslant m} \int_H | D^\alpha u(x)|^p (1 + |x|)^{\lambda|\alpha|/m} \, dx.$$

Summation over those $H \in X$ which meet $G_R \cap \operatorname{supp} u$, and application of Lemma 5, leads to

$$\| u \|^p_{p,G_R} \leqslant \frac{\epsilon^p}{K_0} \sum_{1\leqslant|\alpha|\leqslant m} \int_G | D^\alpha u(x)|^p (1 + |x|)^{\lambda|\alpha|/m} \, dx$$

$$\leqslant \epsilon^p \| u \|^p_{m,p,\lambda}$$

The precompactness in $L^p(G)$ of a sequence $\{u_i\}$ bounded in $W_0^{m,p,\lambda}(G)$ now follows just as in Theorem 1.

(Necessity). Suppose now that G does not satisfy the condition of the theorem. Let X be the partition of E_n constructed in Lemma 4. Then for a sequence $\{H_j\}$ of elements of elements of X and some positive constant K we have

$$C^{m,p}_{H_j,H_j-G} < K.$$

For each such H_j there exists $u_j \in C^\infty(H_j, H_j - G)$ for which

$$\| u \|^p_{p,H_j} = 1, \qquad I^{m,p}_{H_j}(u_j) \leqslant K.$$

Applying Lemma 2 with $\epsilon = 1/2K$ we obtain a $\delta < 1$ for which

$$\| u_j \|^p_{p,\delta H_j} \geqslant 1/2K_2 .$$

By Lemma 3 there exist functions $w_j \in C_0^\infty(H_j)$ such that $w_j(x) = 1$ on δH_j and $| D^\alpha w_j(x)| \leqslant K_3 h^{-|\alpha|}$ for all x, $0 \leqslant |\alpha| \leqslant m$. Let $v_j = u_j w_j \in C_0^\infty(H_j \cap G)$. Then clearly

$$1/2K_2 \leqslant \| v_j \|^p_{p,H_j} \leqslant K_3{}^p.$$

Also for $|\alpha| + |\beta| = m$ we have

$$\int_{H_j} | D^\alpha u_j(x)|^p \, | D^\beta w_j(x)|^p \, (1 + |x|)^\lambda \, dx$$
$$\leqslant K_3{}^p c^{-1} h^{p|\alpha|} \| D^\alpha u_j \|_{p,H_j}^p$$
$$\leqslant K_3{}^p c^{-1} \max(1, K).$$

It follows that $\{v_j\}$ is bounded in $W_0^{m,p,\lambda}(G)$. But since $\| v_j - v_k \|_{p,G}^p = \| v_j \|_{p,H_j}^p + \| v_k \|_{p,H_k}^p \geqslant 1/K_2$ the sequence is not precompact in $L^p(G)$ and so the imbedding $W_0^{m,p,\lambda}(G) \to L^p(G)$ is not compact.

3. Some Geometric Considerations

If $\lambda \leqslant mp$ a domain G for which the imbedding $W_0^{m,p,\lambda}(G) \to L^p(G)$ is compact must satisfy

$$\frac{\operatorname{dist}(x, \operatorname{bdry} G)}{(1 + |x|)^{\lambda/mp}} \to 0$$

as x tends to infinity through G, though, of course, this condition is not in general sufficient to guarantee compactness. The role of the dimension of bdry G on the compactness phenomenon is clarified by the following theorem.

Theorem 3. *Let k be the largest integer satisfying $k < mp$, $k \leqslant n$ (or $k = 1$ if $m = p = 1$). There exists a constant C such that for every cube H in E_n having side $h \leqslant \inf_H (1 + |x|)^\lambda$ we have*

$$C_{H,H-G}^{m,p} \geqslant C \frac{\mu_{n-k}(H, G)}{h^{n-k}},$$

where $\mu_{n-k}(H, G)$ is the maximum, taken over all projections P onto $(n-k)$-dimensional faces of H, of the $(n-k)$ measure of $P(H-G)$.

The proof of Theorem 3 requires the following two lemmas. As both are proved in Ref. [1] we give only sketches of the proofs here.

Lemma 6. *There is a constant C_1 such that for any cube H of side h in E_n, for any subset A of H having positive measure, and for any $u \in C^\infty(H)$*

$$\| u \|_{p,H}^p \leqslant \frac{2^{p-1} h^n}{\operatorname{meas} A} \| u \|_{p,A}^p + C_1 \frac{h^{n+p}}{\operatorname{meas} A} \| \operatorname{grad} u \|_{p,H}^p .$$

Proof. If $x \in H$, $y \in A$ integrate the p-th power of the equation

$$u(x) = u(y) + \int_0^1 \frac{d}{dt} u(y + t(x - y))\, dt$$

first over H and then over A and make use of the formula

$$\int_A \frac{dy}{|x - y|^{n-1}} = c(n)(\text{meas } A)^{1/n}.$$

The result follows.

LEMMA 7. *If $mp > n$ (or $m = p = n = 1$) there is a constant C_2 such that for any cube H of side h in E_n and every $u \in C^\infty(H)$ vanishing in a neighbourhood of some point of H we have*

$$\| u \|_{p,H}^p \leqslant C_2 I_H^{m,p}(u).$$

Proof. If $p \leqslant n$ let $q = np(n - mp + p)^{-1}$; if $p > n$ let $q = p$. Integration of the q-th power of

$$u(x) = \int_0^1 \frac{d}{dt} u(y + t(x - y))\, dt, \qquad u(y) = 0,$$

over H leads to

$$\| u \|_{q,H}^q \leqslant \text{const} \cdot h^q \| \text{grad } u \|_{q,H}^q .$$

The result then follows from Sobolev's imbedding theorem via the technique used in Lemma 1.

Proof of Theorem 3. Let P be the maximal projection referred to in the definition of $\mu_{n-k}(H, G)$ and let $E = P(H - G)$. Without loss of generality we assume that F, the $(n - k)$ face of H containing E, is parallel to the $x_{k+1} \cdots x_n$ coordinate plane. For each point $x = (x', x'')$ in E, where $x' = (x_1, \ldots, x_k)$ and $x'' = (x_{k+1}, \ldots, x_n)$, let $H_{x''}$ be the k cube of side h in which H intersects the k plane through x normal to F. By definition of P there exists $y \in H_{x''} - G$. If $u \in C^\infty(H, H - G)$ then $u(\cdot, x'') \in C^\infty(H_{x''}, y)$. Applying Lemma 7 to $u(\cdot, x'')$ we obtain

$$\int_{H_{x''}} | u(x', x'')|^p \, dx' \leqslant C_2 \sum_{1 \leqslant |\alpha| \leqslant m} h^{p|\alpha|} \int_{H_{x''}} | D^\alpha u(x', x'')|^p \, dx'.$$

Integrating this inequality over E and denoting $H' = \{x' : x = (x', x'') \in H$ for some $x''\}$ we have

$$\| u \|_{p,H'\times E}^p \leqslant C_2 \sum_{1\leqslant|\alpha|\leqslant m} h^{p|\alpha|} \| D^\alpha u \|_{p,H}^p .$$

Application of Lemma 6 with $A = H' \times E$ so that meas $A = h^k \mu_{n-k}(H, G)$ now leads to

$$\begin{aligned} \| u \|_{p,H}^p &\leqslant (2^{p-1}C_2 + C_1) \frac{h^{n-k}}{\mu_{n-k}(H, G)} \sum_{1\leqslant|\alpha|\leqslant m} h^{p|\alpha|} \| D^\alpha u \|_{p,H}^p \\ &= \frac{1}{C} \frac{h^{n-k}}{\mu_{n-k}(H, G)} I_H^{m,p}(u). \end{aligned}$$

Hence

$$C_{H,H-G}^{m,p} \geqslant C \frac{\mu_{n-k}(H, G)}{h^{n-k}} .$$

COROLLARY. *If for given $\epsilon > 0$ there exist R, s with $R \geqslant 0$ and $0 < s \leqslant 1$ and there exists a partition X of E_n into cubes H of side h, where $h^{mp} \leqslant s^m \inf_H(1 + | x |)^\lambda$ such that $\mu_{n-k}(H, G) \geqslant sh^{n-k}/\epsilon$ for all $H \in X$ which meet G_R, then the imbedding $W_0^{m,p,\lambda}(G) \to L^p(G)$ is compact.*

Remark. If $mp > n$, $mp \geqslant \lambda$ we may take $k = n$, $s = \epsilon$. A modification of Lemma 4 assures us that we may find a partition of E_n into cubes satisfying

$$cs^m \sup_H(1 + | x |)^\lambda \leqslant h^{mp} \leqslant s^m \inf_H(1 + | x |)^\lambda.$$

Since $\mu_0(H, G) = 1$ if $H \cap \operatorname{bdry} G \neq \varnothing$ it follows that

$$\frac{\operatorname{dist}(x, \operatorname{bdry} G)}{(1 + | x |)^{\lambda/mp}} \to 0 \qquad \text{as} \quad | x | \to \infty, \quad x \in G$$

is both necessary and sufficient for the compactness of $W_0^{m,p,\lambda}(G) \to L^p(G)$ in this case.

4. An Application to Differential Operators

We consider the $2m$-th order self-adjoint partial differential operator in G given by

$$Lu(x) = (-1)^m \sum_{|\alpha|=m} D^\alpha[a_\alpha(x) D^\alpha u(x)],$$

where the coefficients $a_\alpha \in C^m(\bar{G})$ and satisfy, for some positive constants c_1 and c_2,

$$c_1(1 + | x |)^\lambda \leqslant \operatorname{Re} a_\alpha(x) \leqslant c_2(1 + | x |)^\lambda,$$

and where the Dirichlet form of L,

$$1(u, v) = \sum_{|\alpha|=m} \int_G a_\alpha(x)\, D^\alpha u(x)\, \overline{D^\alpha v(x)}\, dx,$$

satisfies for $u, v \in C_0^\infty(G)$

$$| 1(u, v)| \leqslant c_2 \| u \|_{m,2,\lambda} \| v \|_{m,2,\lambda},$$

$$\operatorname{Re} 1(u, u) \geqslant c_1[\| u \|^2_{m,2,\lambda} - \| u \|^2_{2,G}].$$

The realization in $L^2(G)$ of the operator L corresponding to null Dirichlet boundary data is an operator T defined by

$$\operatorname{dom}(T) = W_0^{m,2,\lambda}(G) \cap \{f \in L^2(G) : Lf \in L^2(G)\},$$
$$Tf = Lf, \qquad f \in \operatorname{dom}(T).$$

For T we have the following standard theorem, a sketch of the proof of which may be found in Ref. [3].

THEOREM 4. *If G is such that the imbedding $W_0^{m,p,\lambda}(G) \to L^p(G)$ is compact, then the operator T as defined above is a closed linear operator in $L^2(G)$; the spectrum $\sigma(T)$ is discrete, has no finite limit points, and is contained in the right half of the complex plane; for $\theta \notin \sigma(T)$ the resolvent operator $R_\theta(T) = (\theta I - T)^{-1}$ is completely continuous.*

REFERENCES

1. R. A. ADAMS, Capacity and compact imbeddings, *J. Math. Mech.*, **19** (1970), 923–929.
2. S. AGMON, "Lectures on Elliptic Boundary Value Problems," Van Nostrand, Princeton, N. J., 1965.
3. C. W. CLARK, An embedding theorem for function spaces, *Pacific J. Math.* **19** (1966), 243–251.